School of Computer Science

60-265-01 Computer Architecture and Digital Design

Fall 2012

Midterm Examination # 1
Tuesday, October 23, 2012

SAMPLE ANSWERS

MARKER COPY

Duration of examination: 75 minutes

1. Answer all questions on this examination paper in the space provided.
2. This is a closed-book examination – no notes or books or electronic computing or storage devices may be used.
3. Do not copy from other students or communicate in any way. All questions from students will be answered only by the attending proctors. There must be absolutely no communication between students during the writing of the examination.
4. The examination must be surrendered immediately when the instructor announces the end of the test period.
5. Each student <u>must</u> sign the examination exit list before leaving the classroom.

Total mark obtained: ___________________

Maximum mark: 41

Question 1. [6 marks]

State the six Boolean Axioms (or Postulates) that are fundamental to Boolean set theory and Boolean Algebra. For each Axiom, state both of the dual forms of algebraic expressions where appropriate. Marks will be deducted for stating any additional theorems that are not axioms.

Axioms/Postulates: (Review the lecture notes on this material) I have provided additional NOTEs along with each postulate. Students are required to provide either a basic statement or set of formulas, or a clear example to illustrate the postulate. Each postulate is very unique and students often do not appreciate the differences and therefore combine two or more and create confusion. Two good questions to ask when confronting new ideas are: (i) what does it mean/imply? AND, (ii) what does it NOT mean/imply? This is the basis for a dualistic approach.

P0: Existence - There exist at least two elements x,y in B such that $x \neq y$

NOTE: If you don't postulate the existence of values/elements, then you have nothing to work with operationally. This does not imply anything else about operations, actual values, etc.

P1: Closure: there exist x,y and operators (+,.) in B such that

$x + y$ in B $x . y$ in B

NOTE: Like the existence of values, there must also be the existence of operators that *act* on the values to transform them. This does not imply what the operators do, only that there are only two of them. This is a necessary condition for the specification for an *algebra*.

P2: Identity: There exist identity elements 0,1 in B relative to the operations + and . , such that for every x in B:

$0 + x = x + 0 = x$ $1 . x = x . 1 = x$

NOTE: The existence of identity elements is meaningful only with respect to their associated operators; hence we must refer to 0 (1) as the identity element with respect to the operator + (.). The choice of symbol (0 or 1 character shapes) is not relevant, nor do these symbols represent a specific quantity of "something" – keep in mind that in practical circuits, both 0 and 1 "values" have a voltage value that is non-zero (zero voltage means the absence of electrical energy, where circuits do not work).

P3: Commutativity: The operations + and . are commutative for all x,y in B:

$x + y = y + x$ $x . y = y . x$

NOTE: It is intriguing that some algebras have a directionality to their operators – these are called *non-commutative algebras*, where X.Y is NOT equal to Y.X. The distinction to permit commutation on both + and . produces a very specialized algebra. Clearly this is not unique to Boolean algebra since we use it all the time with addition and multiplication of decimal and other numbers.

P4: Distributivity: Each operation + and . is distributive over the other; that is, for all x,y,z in B:

$x . (y + z) = x . y + x . z$

$x + (y . z) = (x + y) . (x + z)$

NOTE: These dual forms are very powerful generalizations of the Distribution Law used in ordinary arithmetic and algebra of real numbers where only the first form applies. The extension to the dual form dramatically alters and limits the nature of algebraic expressions and results in significant simplification of otherwise complicated multinomial expressions.

P5: Complementation: For every element x in B there exists an element ~x, called the complement of x, satisfying:

x + ~x = 1   x . ~x = 0

NOTE: The notion of complementation (a generalization of *negation*) is a fundamentally significant idea in Computer Science. As noted here, it may be somewhat easier to state these forms using the ordinary language of sets. Thus, the first form states that, given a portion of a set (x), then creating the *union* of everything else that is not included in x, must yield the complete set with everything in it. This does assume, however, that both x and ~x are included in the full set (a point that is covered above in regard to uniqueness and the specification of the identity relations). The second form states that the *intersection* of a portion x of the set S, together with everything that is not contained in x, namely ~x, is the *empty set*. Strangely enough, this is where the language of set theory is not fully adequate to expose the subtleties of Boolean Set Theory and Algebra because the symbol 0, as the identity element defined over the operator +, is not actually a ***null*** value (ie. empty set). Note also, that both identity values 0 and 1 are associated with their dual operator on the left hand side.

Question 2. [7 marks]

Part 2A. [3 marks]

Simplify the following Boolean SOP expression using Boolean algebra only, expressing your answer in SOP form. Show your steps clearly and completely, but it is not necessary to state which axioms you are using.

F = ABCD + ABC′D′ + A′B′C′D′ + A′BCD′ + ABC′D

+ A′BC′D′ + A′B′C′D + ABCD′ + A′B′CD′

+ AB′C′D + AB′C′D′ + A′BC′D + ABCD

Use idempotency to collect similar terms, then apply simplifications based on X+X′=1.

= AB + C′ + A′CD′

(Question 2 – continued)

Part 2B. [2 marks]
Simplify the following Boolean **SOP** expression using the Karnaugh map technique.

$$F(W,X,Y,Z) = \sum m(0,1,3,5,6,7,8,9,11,13,15)$$

		YZ			
		00	01	11	10
WX	00	1	1	1	0
	01	0	1	1	1
	11	0	1	1	0
	10	1	1	1	0

There is an 8-cube, a 4 –cube (using wraparound) and a 2-cube. The final expression is:

F = Z + X′Y′ + W′XY

Part 2C. [2 marks]
Simplify the following Boolean **POS** expression using the Karnaugh map technique.

$$F(X,Y,Z) = \prod M(0,2,3,4,7)$$

		YZ			
		00	01	11	10
X	0	0	1	0	0
	1	0	1	0	1

There are three 2-cubes. The final expression is:

F = Y′Z′ + YZ + X′Z′

Question 3. [6 marks]

Answer each of the following questions **briefly**. The marks for each question are indicated in square brackets.

(a) Using a truth table, prove that the following Boolean operator relations are equivalent. **[2 marks]**

A xnor B = AB + A′B′

ANSWER TABLE

A	B	A'	B'	AB	A'B'	AB+A'B'	A xnor B
0	0	1	1	0	1	1	1
0	1	1	0	0	0	0	0
1	0	0	1	0	0	0	0
1	1	0	0	1	0	1	1

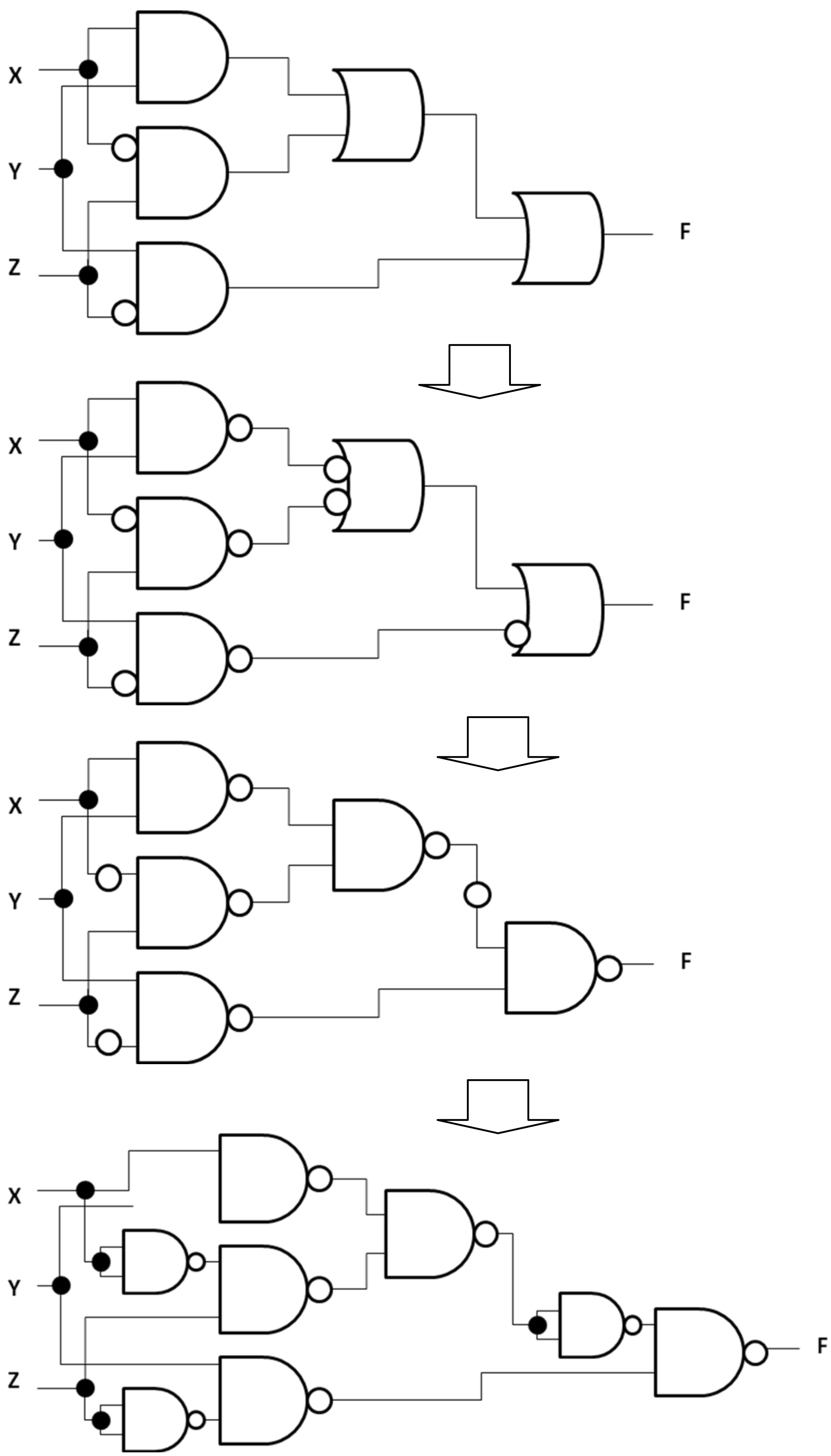
(b) Draw the circuit diagram for the following Boolean expression using only NAND gates. **[2 marks]**

F(X,Y,Z) = XY + X′Z + YZ′

NOTE that AB = (AB)'' = ((AB)')' = ((AB)'(AB)')' = (A nand B) nand (A nand B). Similarly, A+B=(A'B')'=((AA)'(BB)')'=(A nand A) nand (B nand B). Also, A'=(AA)'=A nand A.

F(X,Y,Z) = XY + X′Z + YZ′
= ((XY)′(XY)′)′ + (((XX)′Z)′((XX)′Z)′)′
+ ((Y(ZZ)′)′((Y(ZZ)′)′)′

Students can do the algebraic transformation and then use the expression above to draw the diagram. Alternatively, one can apply diagrammatic transformation rules as shown in the sequence below. To apply these effectively, each student is advised to practice with diagrams while be attendant to the use of various theorems, particularly De Morgan's two dual theorems, and relationships of basic gates to each other, including the NOT, or complementation, operation.

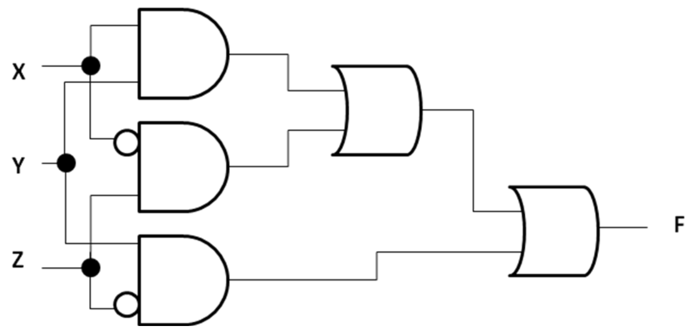

X
Y
Z
F
X
Y
Z
F
X
Y
Z
F
X
Y
Z
F

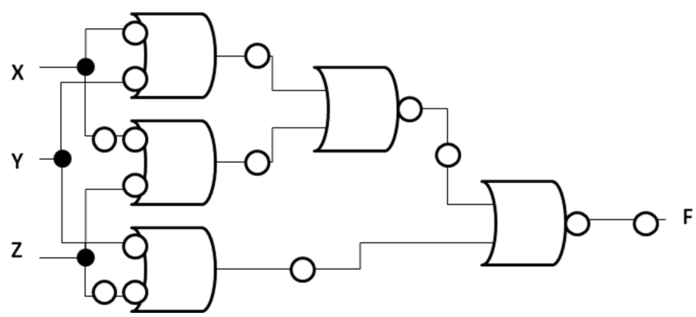
(c) Draw the circuit diagram for the following Boolean expression using only NOR gates. **[2 marks]**

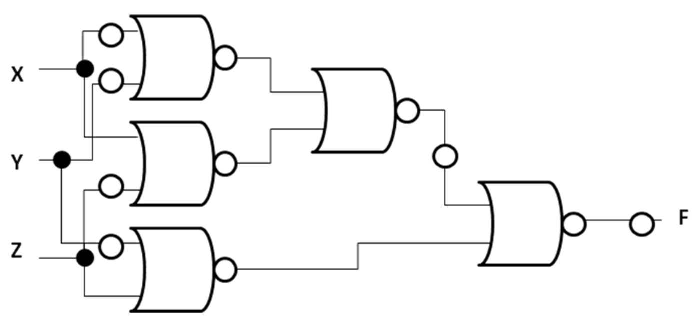
F(X,Y,Z) = XY + X′Z + YZ′

NOTE that A+B=(A+B)''=((A+B)'+(A+B)')'=((A nor B) nor (A nor B)). Also, A'=(A+A)'=A nor A. Also, AB =((AB)')' = (A'+B')' = A' nor B' = (A nor A) nor (B nor B)

As with Part (b), the expression can be first transformed algebraically, then drawn as a circuit. Alternatively, one can perform diagrammatic transformations.







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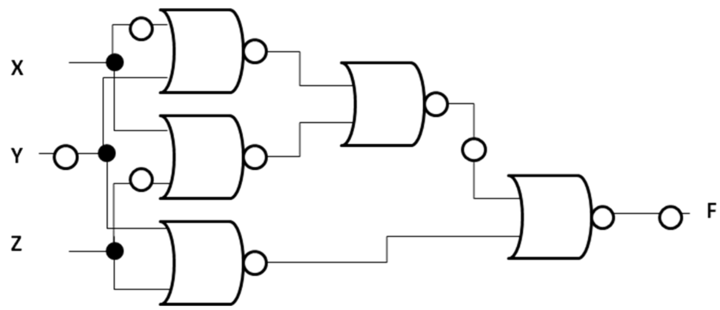
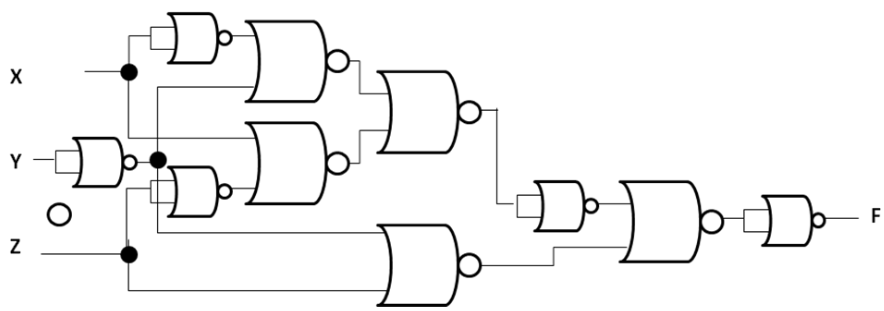

X
Y
Z
F


X
Y
Z
F

Question 4. [10 marks]

Answer each question below. Each part is worth 2 marks. Show your working steps.

A. Convert the decimal number -137_{10} to a signed-binary number using 2's complement and a 16-bit representation. (Note the correction change from 8-bit to 16-bit rep announced during the examination.)

First, convert the absolute value: 137, by successively dividing by 2 and retaining the remainder.

```
2)  137
     68   1
     34   0
     17   0
      8   1
      4   0
      2   0
      1   0
      0   1
Positive ANSWER in 16-bit form:                00000000 10001001
Negative (2's compl) ANSWER in 16-bit form:    11111111 01110111
```

3 Convert the octal number 3655372_8 to a hexadecimal number.

3655372_8 = 011 110 101 101 011 111 010

= 0 1111 0101 1010 1111 1010

= $F5AFA_{16}$

4 Convert the positive decimal real number **0.6** to an unsigned binary number (ie. base-2).

```
0.6 x 2
1.2 x 2
0.4 x 2
0.8 x 2
1.6 (repeats)     ANSWER:   0.1001[1001]
```

5 State the 9's-complement of the 5-digit representation of the decimal (ie. base-10) number: 65432_{10}.

ANSWER: 99999 – 65432 = 34567

(Question 4 – continued)

6 Express the number 4613_9 (ie. base-9) in radix-11 (ie. base-11) form.

$$4613_9 = 4x9^3 + 6x9^2 + 1x9 + 3 = 3414_{10}$$

```
 11 ) 3414
       310   4
        28   2
         2   6
         0   2     ANSWER:   2624₁₁
```

Question 5. [6 marks]

Using Boolean algebra, prove the following theorems. For each step in the proofs you must state the specific axiom or axioms that is/are used. Failure to justify a step will be penalized. Each part is worth 3 marks.

(a) X′+ Y′ = (XY)′ [deMorgan′s Theorem]

Proof: Letting A=(X'+Y') and B=XY, then (a) states: A = B'.
If true, then it must follow that:

A A' = 0	= A B	1. Complementation [P5], Substitution
	= (X'+Y') (XY)	2. Substitution
	= [X' XY] + [Y'XY]	3. Distributivity [P4]
	= [(XX') Y] + [X(YY')]	4. Commutativity [P3]
	= [0.Y] + [X.0]	5. Complementation [P5]
	= 0 + [0. Y] + 0 + [0. X]	6. Identity [P2]
	= Y Y' + [0. Y] + X X' + [0. X]	7. Complementation [P5]
	= Y [Y'+ 0] + X [X' + 0]	8. Distributivity [P4]
	= YY' + XX'	9. Identity [P2]
	= 0 + 0	10. Complementation [P5]
	= 0	11. Identity [P2]

Self-consistent! QED

MARKER NOTE: There may be a tendency after step 5 to reduce terms like 0.Y to 0 directly. This is not correct since it is not using a postulate. This was proven in the lecture slides as Theorem 2, but it still requires its own proof steps based only the postulates. I have combined some steps together where it should be obvious that the same postulate is being used in more than one term within the left side expression. Students may combine other steps together to shorten the proof, but they should still specify what postulates are used. They do NOT need to state the postulate number or even name – they can indicate this by an algebraic statement.

(Question 5 – continued)

(b) A + A′B = A + B

ANSWER:

```
A + A′B = (A+A′)(A+B)     Distributivity
        = 1.(A+B)         Complementation
        = A+B   QED!      Identity
```

Question 6. [6 marks]

Part A. Derive the complete truth table, derive the simplified Boolean expressions and draw the circuit diagram for the Full Adder circuit. This circuit accepts inputs A, B and a C_in (ie. carry in) bit, and produces outputs S (ie. sum) and C_out (ie. a carry out) bits. **[3 marks]**

C_in	A	B	S	C_out
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

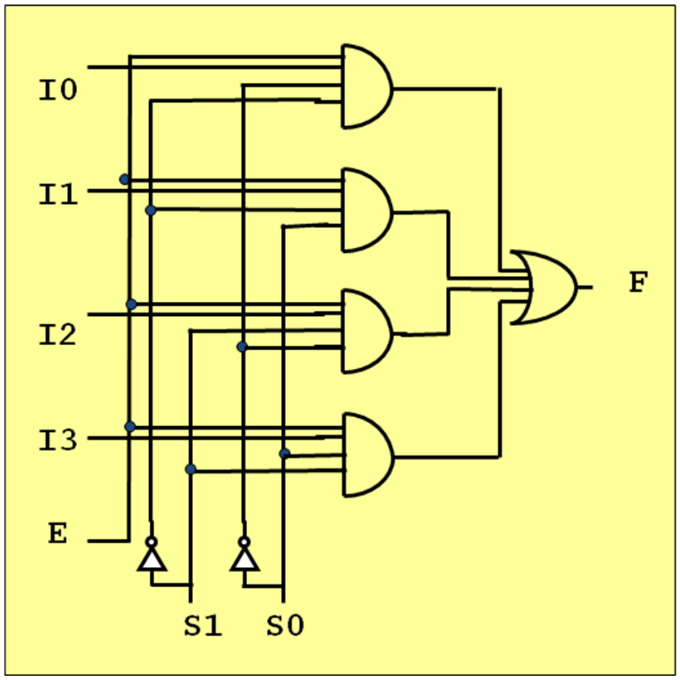
```
S = (A′B + AB′)C_in′ + (AB + A′B′)C_in
  = A xor B xor C_in

C_out = AB + (A xor B)C_in
```

Drawing the circuit is left to an exercise (refer to lecture notes)

(Question 6 – continued)

Part B. Draw the complete circuit diagram for a 4-to-1 line multiplexer. **[3 marks]**

The drawing below is taken directly from the lecture notes. Since students were not asked to provide an Enable input, this can be deleted with no loss of marks.



*** * <u>END OF MIDTERM EXAMINATION</u> * ***